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Gröbner Bases and Normal Forms in a Subring of the Power Series Ring on Countably Many Variables

JAN SNELLMAN
Department of Mathematics, Stockholm University, Sweden

If K is a field, let the ring R' consist of finite sums of homogeneous elements in $R = K[[x_1, x_2, x_3, \ldots]]$. Then, R' contains $\mathcal{M}$, the free semi-group on the countable set of variables $\{x_1, x_2, x_3, \ldots\}$. In this paper, we generalize the notion of admissible order from finitely generated sub-monoids of $\mathcal{M}$ to $\mathcal{M}$ itself; assume that $>$ is such an admissible order on $\mathcal{M}$. We show that we can define leading power products, with respect to $>$, of elements in R', and thus the initial ideal $\mathrm{gr}(I)$ of an arbitrary ideal $I \subset R'$. If I is what we call a locally finitely generated ideal, then we show that $\mathrm{gr}(I)$ is also locally finitely generated; this implies that I has a finite truncated Gröbner basis up to any total degree. We give an example of a finitely generated homogeneous ideal which has a non-finitely generated initial ideal with respect to the lexicographic initial order $>_{lex}$ on $\mathcal{M}$.

1. Introduction

The author was led to the study of Gröbner basis theory of the ring R' when investigating the following problem: What is the initial ideal, in particular, with respect to the lexicographic order, of generic ideals? Recall (Fröberg and Hollman, 1994; Fröberg, 1985; Moreno Socias, 1991) that a generic ideal in a polynomial ring is an ideal generated by generic forms, where furthermore there is no algebraic relation between the coefficients of the generators. When calculating the initial ideals of generic ideals of the same type, but in polynomial rings on successively more variables, one notes that they seem to converge to some monomial ideal in infinitely many variables. It is natural to try to study the initial ideal of the ideal generated by generic forms in infinitely many variables, and try to prove that the sequence of initial ideals indeed converge to this ideal.

In this article, we define the ring R', the natural habitat of generic forms in (countably) infinitely many variables, and prove that we may form initial ideals of, in particular, ideals generated by finitely many generic forms. The fact that this initial ideal may be approximated by the initial ideals of the corresponding ideals in polynomial rings with finitely many variables is the topic of a forthcoming article (Snellman, 1998).

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2. Preliminaries

If S is a ring, and $A \subset S$ is a subset, then $\langle A \rangle_S$ denotes the ideal in S that A generates. Similarly, if M is a monoid, and $A \subset M$ is a subset, then $\langle A \rangle$ denotes the semi-group ideal $\{ma \mid a \in A, m \in M\}$. All rings and monoids under consideration will be commutative.

Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$.

2.1. Power products

Let $\mathcal{N} = \coprod_{\mathbb{N}^+} \mathbb{N}$. For $\boldsymbol{\alpha} \in \mathcal{N}$, a *power product* (or *monomial*) $\boldsymbol{x}^{\boldsymbol{\alpha}}$ in the variables $x_1, x_2, \ldots$ is defined by $\boldsymbol{x}^{\boldsymbol{\alpha}} = \prod_{i=1}^{\infty} x_i^{\alpha_i}$. The set of power products in the variables $x_1, x_2, \ldots$ is a monoid under the obvious multiplication. It is denoted $\mathcal{M} = \{\boldsymbol{x}^{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathcal{N}\}$.

For $\boldsymbol{\alpha} \in \mathcal{N}$, the *total degree* of $\boldsymbol{\alpha}$ is $|\boldsymbol{\alpha}| = \sum_{i=1}^{\infty} \alpha_i$. For a power product $\mathcal{M} \ni m = \boldsymbol{x}^{\boldsymbol{\alpha}}$, the total degree is $|m| = |\boldsymbol{\alpha}|$. The *support* of m is defined by $\mathrm{Supp}(m) = \{i \in \mathbb{N}^+ \mid x_i | m\}$. For $m \neq 1$, this set is non-empty, and has a maximum which is denoted $\mathrm{maxsupp}(m)$, the *maximal support* of m. We use the convention that $\mathrm{maxsupp}(1) = 0$.

For $n \in \mathbb{N}$, define

$$\mathcal{M}^n = \{m \in \mathcal{M} \mid \mathrm{maxsupp}(m) \leq n\}, \qquad \mathcal{M}[n] = \{\boldsymbol{x}^{\boldsymbol{\alpha}} \mid i \leq n \Rightarrow \alpha_i = 0\}.$$

Note that $\mathcal{M}^0$ is the trivial semi-group, and that $\mathcal{M}[0] = \mathcal{M}$. $\mathcal{M}^n$ and $\mathcal{M}[n]$ may be regarded as sub-monoids of $\mathcal{M}$. Furthermore, $\mathcal{M}$ is isomorphic to $\mathcal{M}[n]$ via

$$\mathcal{M} \ni \prod_{i=1}^{\infty} x_i^{\alpha_i} \mapsto \prod_{i=1}^{\infty} x_{i+n}^{\alpha_i} \in \mathcal{M}[n].$$

2.2. The rings R and R'

Let K be a field, and denote by R the ring of power series in countably many variables, with coefficients in K; $R = [[x_1, x_2, \ldots,]]$. For any positive integer n, the power series ring $K[[x_1, \ldots, x_n]]$ is both a subalgebra and a quotient of R, since[†] $\frac{R}{B_n} \simeq K[[x_1, \ldots, x_n]]$, where B_n is the ideal of R generated by all power series in $K[[x_{n+1}, x_{n+2}, x_{n+3}, \ldots]]$ of total degree ≥ 1 and with zero constant term. We define a K-algebra epimorphism ρ_n, the *nth truncation homomorphism*, by means of the composite $R \twoheadrightarrow \frac{R}{B_n} \simeq K[[x_1, \ldots, x_n]]$.

For $n \in \mathbb{N}$, denote by R_n the K-vector space $\{\sum_{\substack{\boldsymbol{\alpha} \in \mathcal{N} \\ |\boldsymbol{\alpha}| = n}} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}\}$. Note that $R_0 = K$, and that $R = \prod_{n \in \mathbb{N}} R_n$. The ring R' is defined as the smallest K-subalgebra of R that contains all homogeneous elements; $R' = \coprod_{n \in \mathbb{N}} R_n$. Note that for $n \in \mathbb{N}^+$, $\rho_n(R') = K[x_1, \ldots, x_n]$. The ring R' is of interest partly because it allows for a generalization of the notion of *generic form*, a generic form in $K[x_1, \ldots, x_n]$ (of some total degree d) being a homogeneous element $f = \sum_{m \in \mathcal{M}^n, |m| = d} c_m m$ where there are no algebraic (over the prime field of K) relations[‡] among the coefficients c_m. In particular, no coefficients belong to the prime field of K, and all c_m's are non-zero. Ideals generated by such elements have been the focus of much study (see, for instance, Fröberg and Hollman (1994); Fröberg (1985)). This definition generalizes directly to R', with f expressed as a (not finite!)

[†] We remark that $x_{n+1} + x_{n+2} + x_{n+3} + \cdots \in \langle K[[x_{n+1}, x_{n+2}, x_{n+3}, \ldots]] \setminus K \rangle_R$ but not in $(x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$, so that $\frac{R}{(x_{n+1}, x_{n+2}, x_{n+3}, \ldots)} \not\simeq K[[x_1, \ldots, x_n]]$.

[‡] Thus, the set $\{c_m \mid m \in \mathcal{M}^n, |m| = k\}$ is *algebraically independent* ("irreduziert", or "algebraische unabhängig") in the sense of Van der Waerden (1930).

linear combination of power products in $\mathcal{M}$ with total degree d. Note that the infinite polynomial ring $K[x_1, x_2, \ldots]$ is not sufficient for this purpose: if f is an element of this ring, then almost every coefficient c_m is zero, which is an element of the prime field. We have that the truncation $\rho_n(f)$ of a generic form in R' is a generic form in $K[x_1, \ldots, x_n]$.

Now let f be an arbitrary, non-zero element of R, $f = \sum_{\boldsymbol{\alpha} \in \mathcal{N}} \boldsymbol{c_\alpha x^\alpha}$. We define the *set of monomials* of f by $\mathrm{Mon}(f) = \{\boldsymbol{x^\alpha} \mid \boldsymbol{c_\alpha} \neq 0\}$, and the *total degree* of f by $|f| = \sup\{|m| \mid m \in \mathrm{Mon}(f)\}$. For $m = \boldsymbol{x^\alpha} \in \mathrm{Mon}(f)$ we define the *coefficient of m in f* by $\mathrm{Coeff}(m, f) = \boldsymbol{c_\alpha}$.

2.3. Admissible orders

Definition 2.1. *By an* admissible order $>$ *on $\mathcal{M}$ we mean a total order such that*

1. $m > 1$ *for all* $m \in \mathcal{M} \setminus \{1\}$.
2. $p > p^{'} \Rightarrow mp > mp^{'}$ *for all* $m, p, p^{'} \in \mathcal{M}$.
3. $x_1 > x_2 > x_3 > \cdots$.

Example 2.2. As an example of an admissible order on $\mathcal{M}$, the *lexicographic* order is defined by $\boldsymbol{x^\alpha} >_{lex} \boldsymbol{x^\beta}$ iff there exist an $n \in \mathbb{N}^+$ such that $\alpha_n > \beta_n$ and for all $k < n$ we have that $\alpha_k = \beta_k$.

Lemma 2.3. *If $n \in \mathbb{N}^+$, $m \in \mathcal{M}^n \setminus \{1\}$ and $p \in \mathcal{M}[n]$, and furthermore $|m| \geq |p|$, then $m > p$ for any admissible order $>$ on $\mathcal{M}$.*

Proof. Denote by V the set $\{x_1, \ldots, x_n\}$ and by W the set $\{x_{n+1}, x_{n+2}, \ldots\}$. Clearly, if $v \in V$ and $w \in W$, then $v > w$. By induction, $\prod_{i=1}^r v_i > \prod_{j=1}^s w_j$ if $r \geq s$.

Now, $m = \prod_{i=1}^r v_i$ with $v_i \in V$ and $r = |m|$. Similarly, $m = \prod_{j=1}^s w_j$ with $w_j \in W$, $s = |p| \leq r$. Therefore, $m > p$. $\square$

If $f \in K[x_1, \ldots, x_n] \setminus \{0\}$ then the set $\mathrm{Mon}(f)$ is finite, and we can find its maximal element, which we call the *leading power product* or *leading monomial* of f. It turns out that R' has the essential property that leading power products can be defined for any non-zero element. Moreover, it can be shown that it is the largest K-subalgebra of R with this property.

Theorem 2.4. *For any admissible order $>$ on $\mathcal{M}$, and any $f \in R' \setminus \{0\}$ the set* $\mathrm{Mon}(f)$ *has a maximal element with respect to $>$.*

Proof. First, assume that the assertion holds for homogeneous elements; then f is a finite sum of its homogeneous components, $f = \sum_{i=0}^{|f|} f_i$, where each $\mathrm{Mon}(f_i)$ has a maximal element p_i. Clearly $\max_{1 \leq i \leq |f|} p_i$ must be maximal also in $\mathrm{Mon}(f)$. Hence, we may assume that f is homogeneous of degree d. Any homogeneous element of degree 1 has a maximal power product; assume inductively that any homogeneous element in R' of degree $< d$ has a maximal power product. Write f in *distributed form* as $f = \sum_{i=1}^\infty x_i g_i$ where $g_i \in R' \cap K[[x_i, x_{i+1}, \ldots]]$. Thus, $x_1 g_1$ contains all terms that are divisible by x_1, and so forth. At least one of the g_i's is non-zero; assume, to simplify the notation, that $g_1 \neq 0$. Since $|g_1| < d$, there exists a maximal power product m_1 of g_1, and $x_1 m_1 \in \mathrm{Mon}(f)$. We claim that any power product $\mathrm{Mon}(f) \ni p > x_1 m_1$ must be divisible by a x_j

with $j < N$, where $N = \text{maxsupp}(m_1)$. To see this, we assume, towards a contradiction, that there exist a monomial $p \in \text{Mon}(f) \cap \mathcal{M}[N]$ such that $p > x_1 m_1$. Since $|p| = |x_1 m_1|$, we get from Lemma 2.3 that $x_1 m_1 > p$, a contradiction.

This shows that the power products of $\text{Mon}(f)$ that precede $x_1 m_1$ are contained in $S = \text{Mon}(\sum_{i=2}^{N} x_i g_i)$. Let us assume that $t \in \text{Mon}(x_j g_j)$, $1 < j \leq N$. It then follows that $t \leq x_j m_j$, where m_j is the maximal power product in $\text{Mon}(g_j)$ (this maximum exists, by the induction hypothesis). Hence, the maximal element of $\{x_2 m_2, \ldots, x_N m_N\}$ is the maximal power product of S.

Therefore, the maximal monomial of $\text{Mon}(f)$ is the maximal element of the finite set $\{x_1 m_1\} \cup \{x_2 m_2, \ldots, x_N m_N\}$. $\square$

Remark 2.5. One can prove the following, stronger statement: suppose that $>$ is a total order on $\mathcal{M}$ which fulfils properties 2.1 and 2.1 of Definition 2.1. Then, every set $\text{Mon}(f)$, when $f \in R'$, has a maximal element w.r.t. $>$ iff every set $\text{Mon}(g)$, where $g \in R'$, $|g| = 1$ has a maximal element w.r.t. $>$.

Definition 2.6. *If $>$ is an admissible order on $\mathcal{M}$, and $f \in R' \setminus \{0\}$, then the* leading power product*, or* leading monomial*, of f is defined by* $\text{Lpp}_>(f) = \text{Lpp}(f) = \max_>(\text{Mon}(f))$. *The* leading coefficient *of f is defined by* $\text{lc}(f) = \text{Coeff}(\text{Lpp}(f), f)$.

Definition 2.7. *For $F \subset R'$,* $\text{in}(F) = \{\text{Lpp}(f) \mid f \in F \setminus \{0\}\}$.

Lemma 2.8. *If I is an ideal (in R'), then $\langle \text{in}(I) \rangle$ is a semi-group ideal in $\mathcal{M}$, and $\langle \text{in}(I) \rangle_{R'}$ is a monomial ideal in R'. The latter ideal is also denoted by* $\text{gr}(I)$.

3. Normal Form Calculations

The calculations of normal forms are an essential and integral part of any Gröbner basis algorithm. To apply these algorithms in the unorthodox setting of the algebra R', we need first to generalize the procedure for finding normal forms. This generalization is also a topic of considerable interest in itself. We will, however, restrict our attention to a narrow class of these normal forms, which, for the purpose of Gröbner basis algorithms, suffices.

3.1. normal form calculations in R'

Remark 3.1. If $t \in \mathcal{M}$, $f \in R'$, $N = \text{maxsupp}(\text{Lpp}(f))$, then $\text{Lpp}(f)|t$ iff $\text{Lpp}(f)|t'$, where t' denotes the sub-word of t that is obtained by replacing any occurrence of variables x_i not in $\{x_1, \ldots, x_N\}$ with 1. So $t = t't''$, with $t' \in \mathcal{M}^N$, $t'' \in \mathcal{M}[N]$.

Similarly, if $F \subset R'$ is a set such that $S = \sup\{\text{maxsupp}(\text{Lpp}(f)) \mid f \in F\}$ is finite (in particular, if F is finite), and if $m \in \mathcal{M}$, then m is divisible by $\text{Lpp}(f)$ for some $f \in F$ iff m' is, where $m' \in \mathcal{M}^S$ denotes the $x_1, \ldots, x_S$ part of m.

It follows from this observation that we, for the purpose of the normal form calculation, may regard R' as a subring of the polynomial ring $K[[x_N, x_{N+1}, \ldots]][x_1, \ldots, x_N]$ since the variables with indices higher than N will "act as coefficients" during the normal form reductions. From now on, unless otherwise stated, we assume that $>$ is some fixed admissible order on $\mathcal{M}$, with respect to which leading power products et cetera are formed.

PROPOSITION 3.2. *Let $F := \{f_1, \ldots, f_r\} \subset R'$ consist of monic elements. For $h \in R'$ there are $h_1, \ldots, h_r, \tilde{h} \in R'$ such that*

$$h = \sum_{i=1}^{r} h_i f_i + \tilde{h}, \quad \mathrm{Lpp}(h_i f_i) \leq \mathrm{Lpp}(h) \quad \textit{and} \quad \tilde{h} = 0 \quad \textit{or} \quad \mathrm{Mon}(\tilde{h}) \cap \langle \mathrm{in}(F) \rangle = \emptyset.$$

We say that $\tilde{h}$ is a "(polynomial) normal form of h with respect to F and $>$".

PROOF. Let $N \geq \max_{1 \leq i \leq r} \mathrm{maxsupp}(\mathrm{Lpp}(f_i))$, that is, $\mathrm{Lpp}(f_i) \in K[x_1, \ldots, x_N]$ for $1 \leq i \leq r$. Consider F as a subset of $K[[x_{N+1}, x_{N+2}, \ldots]][x_1, \ldots, x_n]$ (note that the elements of F are monic there, too). The result then follows from the (well-known) division algorithm for polynomials with coefficients in commutative rings. $\square$

DEFINITION 3.3. *We denote the set of (polynomial) normal forms of h with respect to F by $\mathrm{Norm}_F(h)$. If $0 \in \mathrm{Norm}_F(h)$, then we say that h reduces to zero with respect to F.*

EXAMPLE 3.4. (RALF FRÖBERG, PERSONAL COMMUNICATION.) If $h \in R'$, and $F := \{f_1, \ldots, f_r\} \subset R'$ consists of monic elements, then h may have infinitely many polynomial normal forms with respect to F. Study the normal forms of $h = x_1^2 x_2 (x_3 + x_4 + x_5 + \cdots)$ with respect to $F = \{x_1^2 - x_2 x_3, x_1 x_2 - x_3^2\}$. Regarding R' as a subset of $S_n := K[[x_{n+1}, x_{n+2}, \ldots]][x_1, \ldots, x_n]$ we have that

$$h = \left(\sum_{k=3}^{n} x_1^2 x_2 x_k \right) + x_1^2 x_2 \sum_{k=n+1}^{\infty} x_k, \tag{3.1}$$

The normal forms of $x_1^2 x_2 \sum_{k=n+1}^{\infty} x_k)$ are $\{x_2^2 x_3 \sum_{k=n+1}^{\infty} x_k, x_1 x_3^2 \sum_{k=n+1}^{\infty} x_k\}$. Each of the $n-2$ first terms in (3.1), that is, terms $x_1^2 x_2 x_k$ with $3 \leq k \leq n$, have normal forms in $\{x_2^2 x_3 x_k, x_1 x_3^2 x_k\}$; the resulting terms are linearly independent. Thus, we get normal forms for h by choosing one element from each of the pairs, and adding them. It follows that h has exactly 2^{n-1} different normal forms in S_n, which "lift" to different (polynomial) normal forms in R'.

DEFINITION 3.5. *A non-empty set $F \subset R'$ of homogeneous elements is said to be* locally finite *if $\{f \in F \mid |f| = k\}$ is finite for all k.*

DEFINITION 3.6. *A proper homogeneous ideal I of R' is said to be* locally finitely generated *if*

$$\forall d : \quad \dim_K \frac{I_d}{\sum_{j=1}^{d-1} R'_j I_{d-j}} < \infty.$$

Here, $\sum$ denotes (not direct) sum of K-vector spaces, $I_d = I \cap R_d$, $R'_d = R_d$. Recall that R_d is the set of all homogeneous power series of degree d in R.

LEMMA 3.7. *For a proper homogeneous proper ideal I of R', the following are equivalent:*

1. *I is locally finitely generated.*

2. I has a locally finite generating set.

Proof. If I has a locally finite set of generators F, then F consists of homogeneous elements, and every set $F_t = \{f \in F \mid |f| = t\}$ is finite. Fix a positive integer d. Then

$$I_d = (R'F)_d = \sum_{j=1}^{d} F_j R'_{d-j} = KF_d + \sum_{j=1}^{d-1} F_j R'_{d-j}.$$

Therefore, we can use an noetherian isomorphism (of K-vector spaces) to conclude that

$$KF_d \twoheadrightarrow \frac{KF_d}{KF_d \cap \sum_{j=1}^{d-1} F_j R'_{d-j}} \simeq \frac{KF_d + \sum_{j=1}^{d-1} F_j R'_{d-j}}{\sum_{j=1}^{d-1} F_j R'_{d-j}} = \frac{I_d}{\sum_{j=1}^{d-1} R'_j I_{d-j}}.$$

Since KF_d, by the assumptions, is a finite dimensional K-vector space, we must have that $\dim_K \frac{I_d}{\sum_{j=1}^{d-1} R'_j I_{d-j}} < \infty$.

Conversely, if I is locally finitely generated, we can for each d "lift" a basis of $\frac{I_d}{\sum_{j=1}^{d-1} R'_j I_{d-j}}$ to a finite set $F_d \subset I_d$. Assume by induction that I is generated up to degree $d-1$ by $F_{\leq d-1} = \cup_{i=1}^{d-1} F_i$. We must show that I can be generated up to degree d by $F_{\leq d-1} \cup F_d$. To this end, note that the set

$$T := \{hf \mid h \in R'_j, f \in F_{d-j}, 1 \leq j \leq d-1\}$$

generates the K-vector space $\sum_{j=1}^{d-1} R'_j I_{d-j}$. On the other hand, $\frac{I_d}{\sum_{j=1}^{d-1} R'_j I_{d-j}}$ is finite dimensional, and has a finite basis $\bar{\alpha_1}, \ldots, \bar{\alpha_r}$, which we have lifted to $F_d = \{\alpha_1, \ldots, \alpha_r\} \subset I_d$. It is now an immediate consequence that $KF_d + T$ generates the K-vector space I_d. Therefore, every $h \in I_d$ may be written as

$$h = \sum_{i=i}^{q} f_i h_i + \sum_{j=1}^{r} c_i \alpha_i \qquad f_i \in F_{\leq d-1}, h_i \in R'_{d-|f_i|}, c_i \in K \tag{3.2}$$

This shows that $F_{\leq d-1} \cup F_d$ generates I up to degree d. □

Remark 3.8. In a polynomial ring A, the elements of degree d (of a homogeneous ideal I) that are not generated by elements (in I) of degrees $< d$ correspond to non-zero elements in $\frac{I_d}{A_1 I_{d-1}}$. We can use this simpler expression, because $A_d = A_1 A_{d-1}$ for all d, and hence

$$A_1 I_{d-1} \supset A_2 I_{d-2} = A_1 A_1 I_{d-2} \supset A_3 I_{d-3} = A_2 A_1 I_{d-3} \supset \cdots.$$

For any graded ring, this equality holds if the ring is a polynomial ring over the elements of degree 1; in the literature, one often says that such an A is *naturally graded*.

This condition is *not* fulfilled for the ring R'! To see that, for instance, $R'_1 R'_1 \subsetneq R'_2$, consider the element $\sum_{i=1}^{\infty} x_i^2$, which is not expressable as a *finite sum* of products of linear elements.

Lemma 3.9. *Proposition 3.2 holds when F is locally finite instead of finite, if all the other prerequisites for the theorem are fulfilled.*

Proof. We may assume that h is homogeneous with total degree t. Then h can only be reduced by elements of F with total degree $\leq t$, and we need only consider reductions of h with respect to the finite set of such elements. □

4. Construction of Gröbner Bases

Now that we have developed a satisfactory normal form theory for the algebra R', the construction of Gröbner bases might seem trivial; just do what is done in the polynomial case: start with a finite set of generators, keep adding normal forms of the so-called S-polynomials until no critical pairs remain, and the resulting set will be a Gröbner basis.

There are several difficulties that this, basically sound, method has to overcome. First, we will show that the initial ideal $\mathrm{gr}(I)$ of a finitely generated ideal I of R' need not be finitely generated. Hence, by a Gröbner basis for I we must mean a *possibly infinite* set of generators, whose leading monomials generate $\mathrm{gr}(I)$. It is clear that such a set cannot be calculated in a finite number of steps.

Secondly, to prove that a set of generators is a Gröbner basis, it is customary to show that every element has a unique normal form with respect to it. The normal form theory, developed in the previous part, only deals with normal forms with respect to a finite set, or a locally finite one. Since locally finite sets by definition are homogeneous, the reader might already have guessed how we plan to proceed: we consider only locally finitely generated ideals. Then, starting with a locally finite set of generators, and adding normal forms of S-polynomials, we can arrange things so that we can calculate the Gröbner basis, *up to any given total degree*, in finite time. Since, for an element of degree t, it is only necessary to consider the Gröbner basis up to said degree, we have an algorithm for solving the ideal membership problem.

4.1. Homogeneous Gröbner bases in R'

Definition 4.1. *For $P, Q \in R^{'}$, let the S-polynomial of P and Q be*

$$S(P,Q) = \frac{\mathrm{lc}(Q)\,\mathrm{Lpp}(Q)}{\gcd(\mathrm{Lpp}(P),\mathrm{Lpp}(Q))}P - \frac{\mathrm{lc}(P)\,\mathrm{Lpp}(P)}{\gcd(\mathrm{Lpp}(P),\mathrm{Lpp}(Q))}Q. \tag{4.1}$$

Proposition 4.2. *Let J be a homogeneous ideal in R', and let $F \subset J$ be locally finite (in particular, F consists of homogeneous elements).*

Then the following conditions on F are equivalent:

1. $\langle \mathrm{in}(F) \rangle_{R'} = \mathrm{gr}(J)$,
2. *If* $Q \in J$ *then* $\mathrm{Norm}_F(Q) = \{0\}$,
3. *If* $Q \in J$ *then* $0 \in \mathrm{Norm}_F(Q)$.

If the conditions are fulfilled, then $\langle F \rangle_{R'} = J$.

Proof. It is easy to modify the proofs in Pauer and Pfeifhofer (1988, Proposition 2.5). Note that the authors assume top-reduced normal forms instead of totally reduced normal forms. □

Definition 4.3. *If the conditions of Proposition 4.2 are fulfilled, we say that F is a* Gröbner basis *of J.*

We will need the following results on "partial" or "truncated" Gröbner bases:

PROPOSITION 4.4. *Let J be an homogeneous ideal in R', and let $F \subset J$ be a finite set consisting of homogeneous elements. Let t be a positive integer.*

Then the following conditions on F are equivalent:

1. $(\langle \mathrm{in}(F) \rangle_{R'})_{\leq t} = \mathrm{gr}(J)_{\leq t}$,
2. *If* $Q \in J$, $|Q| \leq t$ *then* $\mathrm{Norm}_F(Q) = \{0\}$,
3. *If* $Q \in J$, $|Q| \leq t$ *then* $0 \in \mathrm{Norm}_F(Q)$.

If the conditions are fulfilled, then $(\langle F \rangle_{R'})_{\leq t} = J_{\leq t}$.

PROOF. The polynomial ring case is treated in Becker and Weispfenning (1993, Theorem 10.39); the generalization to R' is straightforward. □

LEMMA 4.5. *Let J be a (not necessarily homogeneous) ideal in $K[x_1, \ldots, x_n]$, and let $F \subset J$ be a finite set consisting of (not necessarily homogeneous) elements. Let t be a positive integer. Suppose that the admissible order $>$ is* degree-compatible*, that is, $|m| > |m'| \Rightarrow m > m'$. Then the following assertions are equivalent:*

1. $(\langle F \rangle_{K[x_1,\ldots,x_n]})_{\leq t} = \mathrm{gr}(J)_{\leq t}$,
2. *If $P, Q \in J$, $|S(P,Q)| \leq t$ then $0 \in \mathrm{Norm}_F(S(P,Q))$; if $P, Q \in J$, $|S(P,Q)| > t$ then either $0 \in \mathrm{Norm}_F(S(P,Q))$ or all elements of $\mathrm{Norm}_F(S(P,Q))$ have total degree $> t$.*

If the conditions are fulfilled, then $(\langle F \rangle_{K[x_1,\ldots,x_n]})_{\leq t} = J_{\leq t}$.

The following theorem is the main result of this paper:

THEOREM 4.6. *Let I be a homogeneous ideal of R', and let G be a finite set of monic, homogeneous elements in R' that generates I up to degree t. Then, the following assertions are equivalent:*

1. $P, Q \in G$, $|S(P,Q)| \leq t \Rightarrow 0 \in \mathrm{Norm}_G(S(P,Q))$,
2. $\mathrm{gr}(I)_{\leq t} = (\langle \mathrm{in}(G) \rangle_{R'})_{\leq t}$.

It follows that a locally finite set F, consisting of monic elements, is a Gröbner basis of a locally finitely generated ideal J iff every S-polynomial $S(P,Q)$, $P, Q \in F$ reduces to zero with respect to F.

PROOF. (2) $\Rightarrow$ (1): Since $S(P,Q) \in I$, $|S(P,Q)| \leq t$, Proposition 4.4 implies that $0 \in \mathrm{Norm}_G(S(P,Q))$.

(1) $\Rightarrow$ (2): Since I and G are homogeneous, $\mathrm{gr}(I)$ and $\mathrm{in}(G)$ are not changed if we replace the admissible order $>$ with the degree-compatible order $>_{tot}$ defined by $m >_{tot} m'$ if $|m| > |m'|$ or if $|m| = |m'|$ and $m > m'$. We can therefore assume that $>$ is degree-compatible.

It is enough (by induction) to prove the inclusion $\mathrm{gr}(I)_t \subset \langle \mathrm{in}(G) \rangle_{R'}$. Choose a (monic, homogeneous) $h \in I_t \setminus \{0\}$. We must prove that $\mathrm{Lpp}(h) \in \langle \mathrm{in}(G) \rangle_{R'}$.

Let N be the necessary number of "active variables": that is, N indicates which polynomial ring $S_N := K[[x_{N+1}, x_{N+2}, \ldots]][x_1, \ldots, x_N]$ we will embed R' into. We demand four things from N: first, $N \geq \max_{Q \in G} \operatorname{maxsupp}(Q)$, secondly, if $P, Q \in G$ then $N \geq \operatorname{maxsupp}(S(P,Q))$. The third demand is this: we know that for every pair $P, Q \in G$, if the S-polynomial $S(P,Q)$ has total degree $\leq t$, then it reduces to zero with respect to G. Recalling the proof of Proposition 3.2, we obtain that there is some integer n, depending on P and Q, such that the normal form 0 was formed in the polynomial ring S_n. We demand that N is greater than all of these n's, for some choice of normal form reductions to zero of $S(P,Q)$, for every pair $P, Q \in G$ such that $|S(P,Q)| \leq t$.

Since G consists of homogeneous elements, the normal form, with respect to G, of an S-polynomial $S(P,Q)$, $P, Q \in G$, $|S(P,Q)| > t$, is either zero or has total degree $> t$. We demand (the fourth demand) that this is also the case when we "lift" everything to the polynomial ring S_N. If N is too small, then we could have that in the leading power product of the normal form, some variables occuring were regarded as coefficients, which could lower the total degree of the normal form so that it became $\leq t$, resulting in a new minimal monomial generator for the initial ideal of degree $\leq t$. By considering the reductions to normal forms of the finitely many $S(P,Q)$, $P, Q \in G$, $|S(P,Q)| > t$, and choosing sufficiently many "active variables" so that when the reduction chain is regarded as a reduction chain in S_N, the normal form of $S(P,Q)$ (in S_N) always has the same total degree as $S(P,Q)$ (for some choice of a normal form for each S-polynomial), we avoid this calamity.

Injecting S_N into $T_N := K((x_{N+1}, x_{N+2}, \ldots))[x_1, \ldots, x_N]$, where $K((x_{N+1}, x_{N+2}, \ldots))$ is the field of fractions of the domain $K[[x_{N+1}, x_{N+2}, \ldots]]$, we are sure that we can apply standard Gröbner basis techniques. Note that the elements of G are monic even as elements of T_N, so we need never divide with a variable x_j when performing normal form calculations; thus the computations actually take place within S_N. Neither h, the element of $I_t \setminus \{0\}$ choosen above, nor the elements of G need be homogeneous, when regarded as elements of T_N (since some variables get demoted to coefficients when passing from R' to S_N, and therefore homogeneous elements of R' may become non-homogeneous when regarded as elements of S_N), but that is a small matter: the important thing is that the leading power products are preserved. Furthermore, inside T_N, all S-polynomials of degree $\leq t$ reduce to 0 with respect to G. We also have that all S-polynomials of degree $> t$ either reduce to zero or have normal forms with total degree $> t$.

Because of this, the image of G in T_N is a partial Gröbner basis, up to degree t, of the extension of the ideal I to the ideal $I^e \subset T_n$, by Lemma 4.5. It is now clear that when h is regarded as an element of S_N, then $\operatorname{Lpp}(h) \in \langle \operatorname{in}(G) \rangle_{S_N}$. Since N is taken large enough, this implies that when we once more regard h as an element of R', then $\operatorname{Lpp}(h) \in \langle \operatorname{in}(G) \rangle_{R'}$.

The general result follows easily from the result on "partial" Gröbner bases. □

4.2. A Gröbner basis algorithm in R'

The most natural way, perhaps, to extend the usual Gröbner basis algorithm in polynomial rings, is to use the normal form algorithm sketched in 3.2, and try to work directly in R'. That is, we start with a locally finite generating set of our locally finitely generated ideal I, and then proceed, degree by degree, to add normal forms of S-polynomials of the generators; here, the normal forms are elements in R'.

We can also work within the polynomial rings $K((x_{n+1}, x_{n+2}, \ldots))[x_1, \ldots, x_n]$, successively promoting “constants” to “variables” as the need arises. The resulting algorithm would not differ from the one we describe; it is merely another way of viewing the original one. In Appendix A.1 we sometimes take this view when we talk about “splitting the coefficients” and “active variables”.

In either case, the algorithm works with homogeneous in-data, and uses a variant of the so called *normal selection strategy* as defined in Buchberger (1979) and Gebauer and Möller (1988); it uses this strategy, but the admissible order $>_{tot}$ defined by $m >_{tot} p$ iff $|m| > |p| \vee (|m| = |p| \wedge m > p)$ is used for comparisons. Note that every element in the (preliminary) Gröbner basis will be homogeneous, and hence that every comparison of monomials will, in fact, compare monomials of the same total degree, for which $>$ and $>_{tot}$ coincide. So, the run of the Gröbner basis algorithm, and hence the result, is not changed if we replace $>$ with $>_{tot}$ throughout.

We recall that the normal selection strategy chooses the critical pair (P, Q) with the least $\mathrm{lcm}(\mathrm{Lpp}(P), \mathrm{Lpp}(Q))$. In particular it adds the S-polynomial with lowest total degree first. This is essential, since it guarantees that after each step of the algorithm, the partial Gröbner basis is a locally finite set, and that we, for any total degree t, can compute all elements of the Gröbner basis with total degree $\leq t$ in a “finite number of steps” (thus yielding a solution to the ideal membership problem); however, each “step” involves a complicated normal form calculation. In fact, even the seemingly innocuous operation of forming S-polynomials involves infinite operations. Hence, we are not sure that it can be computed in finite time (with, for instance, a Turing machine). Furthermore, we have not placed any restrictions on the field K; it may not be “effectively computable”, a technical condition *not* fulfilled for such commonplace rings as $\mathbb{R}$ and $\mathbb{C}$. More on this matter may be found in Seidenberg (1974).

To continue with the description of the “algorithm”: we add normal forms of S-polynomials as generators, and the normal form sets with respect to the partial Gröbner basis need not be singletons. Therefore, we need to make another choice: what normal form to add. We will tacitly assume the existence of some suitable choice function to facilitate this.

A final remark: the so called Buchberger Criteria can, appropriately modified, be also used in this “algorithm” to avoid unnecessary reductions of S-polynomials.

Remark 4.7. If $C = (P, Q) \in G$ is a critical pair of elements in F, then if the Gröbner basis elements P and Q are changed (as a result of a reduction with respect to a new Gröbner basis element) then the corresponding constituent of C is implicitly assumed to change accordingly. Thus, in a practical implementation, one would save the *pair of indices* of the Gröbner basis elements, rather than the elements themselves.

Algorithm 4.1.
Specification: $F := \mathrm{GBAS}(\{f_1, f_2, f_3, \ldots\})$
Construction of standard basis F of $\langle\{f_1, f_2, f_3, \ldots\}\rangle_{R'}$
Given: A locally finite generating set $\{f_1, f_2, f_3, \ldots\} \subset R^{'}$,
homogeneous with $\mathrm{Lpp}(f_i) = m_i$.
Find: $F = \cup_{g=1}^{\infty} F_g$, a locally finite set which is a Gröbner basis for $\langle\{f_1, \ldots, f_r\}\rangle_{R'}$.
Variables:
F_i = The Gröbner basis elements of total degree i
G_i = Critical pairs which have S-polynomial of total degree i.
$F = \cup_{i>0} F_i$ at all times

$G = \cup_{i>0} G_i$ at all times

for $g := 1 \ldots \infty$
  while $G_g \neq \emptyset$
    Choose a pair $(P, Q) \in G_g$ according to the normal selection strategy (using $>_{tot}$)
    $G_g := G_g \setminus (P, Q)$
    if $0 \notin \mathrm{Norm}_F(S(P, Q))$
      Choose $h \in \mathrm{Norm}_F(S(P, Q)) \subset R'$
      $h := \frac{h}{\mathrm{lc}(h)}$
      reduce F_g with respect to h
      $F_g := F_g \cup \{h\}$
      forall $W \in F \setminus \{h\}$
        $d := |\,\mathrm{lcm}(\mathrm{Lpp}(W), \mathrm{Lpp}(h))|$
        $G_d := G_d \cup \{(W, h)\}$
      end for
    end if
  end while
  forall $f \in \{f_i \mid |f_i| = g\}$
    if $0 \notin \mathrm{Norm}_F(h)$
      Choose $h \in \mathrm{Norm}_F(f)$
      $h := \frac{h}{\mathrm{lc}(h)}$
      Reduce F_g with respect to h
      $F_g := F_g \cup \{h\}$
      forall $W \in F \setminus \{h\}$
        $d := |\,\mathrm{lcm}(\mathrm{Lpp}(W), \mathrm{Lpp}(h))|$
        $G_d := G_d \cup \{(W, h)\}$
      end for
    end if
  end for
end for

It is an easy consequence of the previous results, that the output of Algorithm 4.1 is indeed a Gröbner basis:

THEOREM 4.8. *Let I be a homogeneous ideal in R', generated by a locally finite set $\{f_1, f_2, f_3, \ldots\}$ (thus, I is locally finitely generated). If $F = \cup_{g=1}^{\infty} F_g$ is the output of Algorithm 4.1, then F is a Gröbner basis of I. Since F is a locally finite set, so is the set $\{\mathrm{Lpp}(f) \mid f \in F\}$, which generates* $\mathrm{gr}(I)$*. Therefore,* $\mathrm{gr}(I)$ *is locally finitely generated.*

REMARK 4.9. One can easily prove that F has most of the usual properties of a Gröbner basis in a polynomial ring (see Becker and Weispfenning (1993) and Buchberger (1985a)) so that, for instance, normal forms with respect to F are unique. However, it is impossible to decompose the K-vector space R' as $R' = I \oplus \mathrm{Span}(\mathcal{M} \setminus \mathrm{gr}(I))$. This follows from the fact that $\mathrm{Span}(\mathcal{M}) = K[x_1, x_2, x_3, \ldots] \subsetneq R'$.

Appendix A. Examples of Lexicographic Initial Ideals of Generic Ideals

A.1. a finitely generated initial ideal: two generic quadratic forms

In this section, we will calculate the initial ideal (with respect to the lexicographic order) of the generic ideal spanned by two generic quadratic forms. By "generic ideal", we mean, as in Fröberg and Hollman (1994) and Fröberg (1985), that not only are the generators generic, but they are independent in the sense that the union of their sets of coefficients is algebraically independent. Let therefore $I = (f_1, f_2)$ where $f_1, f_2 \in R_2$ have generic coefficients. There should be no algebraic relation among the non-zero coefficients, nor should these belong to the prime field of K. To avoid complicating matters, we will in fact assume that $K = \mathbb{C}$ with prime field $\mathbb{Q}$.

To facilitate computations, we perform a "Gaussian-elimination" step and write the generators as

$$\begin{aligned} f_1 &= {x_1}^2 + a_{1,3}x_1 + \alpha_{2,2}{x_2}^2 + a_{2,3}x_2 + a_3 \\ f_2 &= x_1x_2 + b_{1,3}x_1 + \beta_{2,2}{x_2}^2 + b_{2,3}x_2 + b_3 \end{aligned}$$

where $a_{1,3} = \sum_{j=3}^{\infty} \alpha_{1,3}x_j$, $a_{2,3} = \sum_{j=3}^{\infty} \alpha_{2,3}x_j$, $a_3 = \sum_{3\le i\le j} \alpha_{i,j}x_ix_j$, $b_{1,3} = \sum_{j=3}^{\infty} \beta_{1,3}x_j$, $b_{2,3} = \sum_{j=3}^{\infty} \beta_{2,3}x_j$ and $b_3 = \sum_{3\le i\le j} \beta_{i,j}x_ix_j$. Following the algorithm, we regard the f_i as elements in $K[[x_3, x_4, \ldots]][x_1, x_2]$ and form the S-polynomial:

$$\begin{aligned} S_{1,2} &= x_2f_1 - x_1f_2 \\ &= -b_{1,3}{x_1}^2 - \beta_{2,2}x_1{x_2}^2 + (a_{1,3} - b_{2,3})x_1x_2 - b_3x_1 + \alpha_{2,2}{x_2}^3 + a_{2,3}{x_2}^2 + a_3x_2. \end{aligned}$$

When we reduce this to normal form, the leading monomial is $(-\beta_{2,2}{\beta_{1,3}}^2 + \beta_{1,3}\beta_{2,3} - \beta_{3,3})x_1x_3^2$. Thus, for the next step of the algorithm we need to add x_3 as an active variable. In $K[[x_4, \ldots]][x_1, x_2, x_3]$ the generators can be written as

$$\begin{aligned} f_1 &= {x_1}^2 + \alpha_{1,3}x_1x_3 + a_{1,4}x_1 + \alpha_{2,2}{x_2}^2 + \alpha_{2,3}x_2x_3 + a_{2,4}x_2 + \alpha_{3,3}{x_3}^2 + a_{3,4}x_3 + a_4 \\ f_2 &= x_1x_2 + \beta_{1,3}x_1x_3 + b_{1,4}x_1 + \beta_{2,2}{x_2}^2 + \beta_{2,3}x_2x_3 + b_{2,4}x_2 + \beta_{3,3}{x_3}^2 + b_{3,4}x_3 + b_4 \\ f_3 &= x_1{x_3}^2 + Q \end{aligned}$$

where $a_{1,4} = \sum_{j=4}^{\infty} \alpha_{1,4}x_j$, $a_{2,4} = \sum_{j=4}^{\infty} \alpha_{2,4}x_j$, $a_{3,4} = \sum_{j=4}^{\infty} \alpha_{3,4}x_j$, $a_4 = \sum_{4\le i\le j} \alpha_{i,j}x_ix_j$, $b_{1,4} = \sum_{j=4}^{\infty} \beta_{1,4}x_j$, $b_{2,4} = \sum_{j=4}^{\infty} \beta_{2,4}x_j$, $b_{3,4} = \sum_{j=4}^{\infty} \beta_{3,4}x_j$, $b_4 = \sum_{4\le i\le j} \beta_{i,j}x_ix_j$ and Q is a rather longish expression that is omitted in the interest of brevity. Now we form the S-polynomial of f_1 and f_3 in $K[[x_4, \ldots]][x_1, x_2, x_3]$ and reduce it with respect to $\{f_1, f_2, f_3\}$. The resulting expression is somewhat long, so we give here only the leading term, which is $-\frac{\beta_{2,2}\left({\beta_{2,2}}^2+\alpha_{2,2}\right)}{\beta_{2,2}{\beta_{1,3}}^2-\beta_{1,3}\beta_{2,3}+\beta_{3,3}}x_2^4$. Since the leading coefficient lies in K, we need not split the coefficients. We add f_4, a monic polynomial in $K[[x_4, \ldots]][x_1, x_2, x_3]$ with leading monomial x_2^4, to our basis. Forming $S(f_2, f_3)$, we find that it reduces to 0 with respect to $\{f_1, f_2, f_3, f_4\}$. We are now done, since $S(f_i, f_4)$ must , for $i = 1, 3$, reduce to 0 with respect to $\{f_1, f_2, f_3, f_4\}$ by *Buchbergers first criterion*, and $S(f_2, f_4)$ reduce to 0 as well. Lifting the result back to R', we have that $\mathrm{gr}(I) = (x_1^2, x_1x_2, x_1x_3^2, x_2^4)$.

A.2. a finitely generated ideal having non-finitely generated initial ideal: the generic ideal generated by a quadratic and a cubic form

If we modify the previous example, studying the generic ideal $I = (f, g)$ where f is a quadratic generic form and g is a cubic generic form, then, the (lexicographic) initial

Table 1. Initial ideals of restricted ideals of the generic ideal generated by a quadratic and a cubic form.

Degree	$\mathrm{gr}(\rho_2\,(I))$	$\mathrm{gr}(\rho_3\,(I))$	$\mathrm{gr}(\rho_4\,(I))$	$\mathrm{gr}(\rho_5\,(I))$	$\mathrm{gr}(\rho_6\,(I))$	$\mathrm{gr}(\rho_7\,(I))$
2	x_1^2	x_1^2	x_1^2	x_1^2	x_1^2	x_1^2
3	$x_1x_2^2$	$x_1x_2^2$	$x_1x_2^2$	$x_1x_2^2$	$x_1x_2^2$	$x_1x_2^2$
4	x_2^4	$x_1x_2x_3^2$	$x_1x_2x_3^2$	$x_1x_2x_3^2$	$x_1x_2x_3^2$	$x_1x_2x_3^2$
5		$x_1x_3^4$	$x_1x_2x_3x_4^2$	$x_1x_2x_3x_4^2$	$x_1x_2x_3x_4^2$	$x_1x_2x_3x_4^2$
6			$x_1x_2x_4^4$	$x_1x_2x_3x_4x_5^2$	$x_1x_2x_3x_4x_5^2$	$x_1x_2x_3x_4x_5^2$
6		x_2^6	x_2^6	x_2^6	x_2^6	x_2^6
7				$x_1x_2x_3x_5^4$	$x_1x_2x_3x_4x_5x_6^2$	$x_1x_2x_3x_4x_5x_6^2$
7			$x_1x_3^6$	$x_1x_3^6$	$x_1x_3^6$	$x_1x_3^6$
8					$x_1x_2x_3x_4x_6^4$	$x_1x_2x_3x_4x_5x_6x_7^2$
8				$x_1x_2x_4^6$	$x_1x_2x_4^6$	$x_1x_2x_4^6$
9						$x_1x_2x_3x_4x_5x_7^4$
9					$x_1x_2x_3x_5^6$	$x_1x_2x_3x_5^6$
10						
10						$x_1x_2x_3x_4x_6^6$

ideal $\mathrm{gr}(I)$ is locally finitely generated but not finitely generated. In fact, the initial ideal $\mathrm{gr}(I)$ is generated by

$$x_1^2,\ x_1x_2^2,\ x_1x_2x_3^2,\ x_1x_2x_3x_4^2,\ x_1x_2x_3x_4x_5^2,\ x_2^6,$$
$$x_1x_2x_3x_4x_5^2,\ x_1x_3^6,\ x_1x_2x_3x_4x_5x_6^2,\ x_1x_2x_4^6,\ x_1x_2x_3x_4x_5x_6x_7^2,\dots$$

where, for a total degree $t \geq 9$, the minimal monomial generators of degree t are $x_1x_2\cdots x_{t-6}x_{t-4}^6$ and $x_1x_2\cdots x_{t-1}x_t^2$. This initial ideal provides some information on the initial ideals of the restricted ideals $\rho_n\,(I) \subset K[x_1,\dots,x_n]$ of I: these are ordinary generic ideals generated by a quadratic and a cubic form. Their initial ideals have been studied by Alyson Reeves (1993). We tabulate the first of these initial ideals in Table 1.

The author has proved (Snellman, 1998), that, for all locally finitely generated ideals J, the relation $\lim_{n\to\infty}\mathrm{gr}(\rho_n\,(J)) = \mathrm{gr}(J)$ holds, in the following sense:

$$\forall d : \exists N(d) : n > N(d) \Rightarrow \mathrm{gr}(J)_{\leq d} = (\langle \mathrm{gr}(\rho_n\,(J))\rangle_{R'})_{\leq d}.$$

So the initial ideals of all restricted ideals determine $\mathrm{gr}(J)$; the converse, on the other hand, does not hold in general: studying Table 1, we see that $\mathrm{gr}(\rho_2\,(I))$ has the minimal monomial generator x_2^4; this "tail", which may be regarded as an effect of the truncation to two variables (the corresponding generator of the same degree for $\mathrm{gr}(I)$ is $x_1x_2x_3^2$) is impossible to detect from the study of $\mathrm{gr}(I)$ alone.

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